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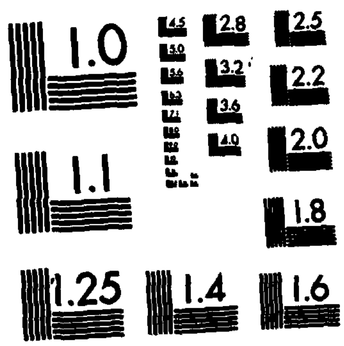

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This report addresses the extension of present robust Wiener filtering theory to multivariable dynamical systems. By considering only linear systems with multiple state processes and a single observation process, two reasonable formulations are posed for solution. Methods are presented for finding solutions, and for non-causal problems with spectral-band noise uncertainty classes the two formulations are shown to yield identical solutions. A simple design example illustrates the procedure for a double-integrator plant.

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by

John Charles Darragh

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ROBUST STATE ESTIMATION FOR
MULTIPLE-STATE SYSTEMS

by

JOHN CHARLES DARRAGH

B.S., University of Illinois, 1980

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Electrical Engineering
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1. INTRODUCTION

The study of optimal filtering has evolved around the well-known techniques of Kalman filtering [1] and Wiener filtering [2]. Though each provides a convenient method of designing optimal filters, both have inherent limitations on the nature of the system to which they may be applied. In particular, Kalman filtering requires the noise to be white or representable as the output of a finite-dimensional linear system driven by white noise. The Wiener approach allows for wide sense stationary second-order noise processes with second-order properties that are known exactly. For many practical situations precise knowledge of these second-order properties is unrealistic. Vastola [3] has shown that the performance degradation of Wiener filters under small deviations from assumed nominal noise spectra can render such a design virtually useless upon implementation.

Consequently, a preferable design procedure might guarantee a minimum performance level over a range of power spectral density functions in the noise process. Several authors have approached this robust Wiener filtering problem using the game-theoretic minimax formulation and a mean-square error cost function, under various assumptions on the filter structure and models by which the classes of power spectral densities (PSD's) are specified. Kassam and Lim [4] have solved the non-causal robust filtering problem for two useful spectral uncertainty classes, known as the spectral-band and mixture (or ε-band) uncertainty models. Poor [5] has solved the "total-variation" non-causal problem, and in the same paper addressed some aspects of the causal problem. Poor and Looze [6] have also discussed in some detail the solution of causal problems for systems which

may be represented by single-dimensional linear, time-invariant state-space equations with either process or observation noise known to be white and the other process specified by a spectral uncertainty class. The common feature of all of these studies is that they are limited to problems in which a single state process must be estimated from a single observation process. Therefore these robust filtering results are applicable only to inherently "single-dimensional" estimation problems.

The usefulness of the robust filtering ideas could be enhanced considerably if they were extended to estimation of vector state processes from vector observation processes. The author is unaware of any robust filter results for this most general multivariable problem, aside from the special case of vector white process and observation noises with uncertain componentwise correlation treated by Poor and Looze [6].

This thesis represents a first investigation of how robust filtering problems may be formulated and solved in the multivariable setting. Attention is focused on problems of estimating a state vector for dynamical systems driven by a scalar process noise, using a scalar observation process corrupted by a scalar observation noise. For these multiple-state problems, two suitable robust filter problem formulations are posed, the state-by-state and least-squares robust problems. Solution of the state-by-state problem follows easily from one-dimensional techniques; the same is not true of the least-squares formulation.

However, when attention is restricted to non-causal problems with spectral-band noise uncertainties, the state-by-state and least-squares approaches are shown to yield the same solution. This equivalence is established first as a theorem supported by the state-by-state solution

procedure and properties of the minimax formulations, and secondly by direct calculation of the least-squares solution through an appropriate transformation. Consequently, design of robust filters for state estimation (either state-by-state or lease-squares robust) may be performed using the more expedient state-by-state methodology. Accordingly, a design procedure is outlined for non-causal, spectral band, multiple-state robust estimators.

Many practical state estimation problems require a causal estimator structure. Unfortunately, the non-causal techniques are not generally applicable to causal problems. Some aspects of the causal estimation problem are discussed, since they illustrate the true multivariable character of the general state-by-state and least-squares approaches and reinforce the significance of the non-causal findings.

Section 2 is intended to provide a brief review of the optimal filtering problem and summarize relevant robust one-dimensional Wiener filtering results. Section 3 covers multiple-state robust estimation. Section 4 illustrates the significant results of Section 3 with a design example. Finally, Section 5 summarizes the findings of this paper and poses several issues for further study.

2. OPTIMAL ESTIMATION

This chapter provides the requisite background information for the subsequent discussion of multiple-state estimation. The first section outlines the well-known results of optimal filtering for continuous-time systems. Stationary, linear, time-invariant Wiener filtering is developed and the connection to the popular methods of Kalman drawn. The remainder of this chapter is devoted to discussing robust state estimation results for one-dimensional systems. The minimax problem is posed, the relationship to the associated maximin problem discussed, and suitable spectral uncertainty classes presented. The specific example of spectral-band uncertainty for non-causal filtering is elaborated on for use in subsequent chapters.

2.1. Solutions to the Wiener-Hopf Equation

This thesis is concerned with estimating a continuous-time state process denoted x_t (or equivalently $x(t)$) from a continuous-time observed process, y_τ ($y(\tau)$). Each process may, in general, be an infinite-dimensional vector process. The estimation problem is then to obtain $\hat{x}_t$, an estimate of the state process which satisfies

$$\hat{x}_t = E\{x_t \mid y_\tau\}, \qquad \tau \in \mathcal{J}_y \;, \; t \in \mathcal{J}_x . \tag{2.1.1}$$

The observation process may be represented by an equation

$$y_\tau = \mathcal{H}[x_t] + n_\tau \quad , \; t \in \mathcal{J}_x \quad \tau \in \mathcal{J}_y \;, \tag{2.1.2}$$

with $\mathcal{H}(\cdot)$ indicating any mean-square integrable function of x_t. It may be assumed that the processes $\mathcal{H}[x_t]$, n_τ and y_τ are second-order (i.e. $E\{n^2\} < \infty$

for all $\tau \in \mathcal{J}_y$, etc.), that the second-order properties of each process are known, and that the optimal filter is defined as the filter which minimizes the weighted mean-square error beteeen x_t and $\hat{x}_t$, then the problem may be formulated using the techniques of Wiener and Kolmogorov. The optimization problem may then be stated

$$\hat{x}_t = \underset{\hat{x}_t \in \mathcal{H}_y}{\arg\min}\ E\{(x_t-\hat{x}_t)^T Q(x_t-\hat{x}_t)|y_\tau\} \qquad \text{for all } t\in\mathcal{J}_x\ ,\ \tau\in\mathcal{J}_y, \tag{2.1.3}$$

and finite-dimensional x_t and y_τ ; where Q is an n×n nonnegative definite symmetric weighting matrix (assuming x_t is an n×1 vector), and $\mathcal{H}_y$ denotes the class of all mean-square integrable functions of y_τ, t and τ. Elementary mean-square estimation arguments (see for example, Papoulis [7]) show that the optimal mean-square estimator is affine in y_τ (of the form $\hat{x}_t = \mathcal{L}(y_\tau)+b$, with $\mathcal{L}$ a linear operator). Hence, without loss of generality, $\mathcal{H}_y$ may be restricted to affine functions of y_τ. In fact the orthogonality principle (see for example, Wong [8]) states that

$$\hat{x}_t = \int_a^b h_{opt}(t,\tau)y(\tau)d\tau + b \tag{2.1.4}$$

solves the minimization problem (2.1.3) if and only if

$$E\{x_t\} = E\{\hat{x}_t\} \tag{2.1.5}$$

and

$$E\ (x_t-x_t)(y\ )^T = 0, \qquad t\in\mathcal{J}_x\ ,\ \ \tau\in\mathcal{J}_y = [a,b]. \tag{2.1.6}$$

The first condition dictat s that the mean function of x_t satisfy

$$E\{x_t\} = E\{\hat{x}_t\} = \int_a^b h_{opt}(t,\tau)E\{y_\tau\}d\tau + b, \tag{2.1.7}$$

and the second implies

$$R_{xy}(t,\tau) = \int_a^b h_{opt}(t,\sigma)R_y(\sigma,\tau)d\sigma + bE\{y_\tau\}, \tag{2.1.8}$$

where

$$R_{xy}(t,\tau) = E\{x_t\ y_\tau^T\} \text{ and } R_y(\sigma,\tau) = E\{y_\sigma\ y_\tau^T\}. \tag{2.1.9}$$

$h_{opt}(t,\tau)$ will be termed the optimal impulse response. Note that since the covariance matrix $C_{xy}(t,\tau)$ may be expressed as

$$C_{xy}(t,\tau) = R_{xy}(t,\tau) - E\{x_t\}E\{y_\tau^T\} \tag{2.1.10}$$

and from (2.1.7) above

$$b = E\{x_t\} - \int_a^b h_{opt}(t,\tau)E\{y_\tau\}d\tau, \tag{2.1.11}$$

equation (2.1.8) reduces to

$$C_{xy}(t,\tau) = \int_a^b h_{opt}(t,\sigma)C_y(\sigma,\tau)d\sigma. \tag{2.1.12}$$

Without loss of generality, it may be assumed that

$$E\{x_t\} = E\{y_\tau\} = 0. \tag{2.1.13}$$

Consequently

$$R_{xy}(t,\tau) = \int_a^b h_{opt}(t,\sigma)R_y(\sigma,\tau)d\sigma, \qquad \sigma \in [a,b]. \tag{2.1.14}$$

This fundamental result is commonly referred to as the Wiener-Hopf equation (for continuous time processes). When a solution can be found, the optimal estimate is given by

$$\hat{x}_t = \int_a^b h_{opt}(t,\tau)y(\tau)d\tau. \tag{2.1.15}$$

This is the Wiener-Hopf formulation of the optimal filtering problem in a quite general form. The underlying assumptions made to pose a problem in this form are:

i) The processes involved are second-order, with known second order properties. That is, the mean and auto-correlation of each process may be calculated from the problem statement. Additionally, the cross-correlation between processes x_t and y_τ is also given either explicitly or implicitly.

ii) The optimal filter is the best linear filter or (equivalently) the minimum mean-square error filter.

iii) The state, noise and observation processes are all continuous-time vector quantities of finite dimension.

Now it would appear that the optimal filtering problem is solved. Indeed equation (2.1.14) may be solved numerically in its most general form to yield an approximate solution useful for some purposes (i.e. design of Wiener filters for systems with well-known spectral properties). However, for purposes of the subsequent discussion of robust filtering, more explicit expressions for $h_{opt}(t,\tau)$ are highly desirable. Such explicit expressions exist for several special cases of the more general problem statement.

Perhaps the most immediate result occurs if all of the random processes are not only second order, but wide-sense stationary as well, and $\mathcal{J}_y = (-\infty, +\infty) = (a,b)$. Such an estimator is non-causal since the entire sample path of y_τ is used to estimate x_{t_1} for each $t_1 \in \mathcal{J}_x$; the observed

process after t_1 is used to compute $\hat{x}(t_1)$. Since $R_y(\tau,\sigma)$ then equals $R_y(\tau-\sigma)$, then Wiener-Hopf equation becomes

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} h_{opt}(\sigma) R_{\tau}(\tau-\sigma)d\sigma. \tag{2.1.16}$$

The time-invariance and infinite bounds on the integral make solution routine using Fourier transform techniques, resulting in the explicit solution

$$H_{opt}(\omega) = S_{xy}(\omega)[S_y(\omega)]^{-1}. \tag{2.1.17}$$

The optimal filter is then of the form

$$\hat{x}(\omega) = H_{opt}(\omega)Y(\omega) \tag{2.1.18}$$

or in the time domain again

$$\hat{x}_t = \int_{-\infty}^{\infty} h_{opt}(\sigma)y(\tau-\sigma)d\sigma. \tag{2.1.19}$$

Of course $S_{xy}(\omega)$ denotes the cross spectral density of x_t and y_t (the Fourier transform of $R_{xy}(\tau)$), and capital letters denote the Fourier transforms of their corresponding lower-case counterparts. This result is very significant in that it provides a very easily computed optimal filter when a problem may be posed in such a form. Equation (2.1.17) will be used extensively in the following chapters.

A second special case of the Wiener-Hopf equation arises if the estimate of $x(t_1)$ is required at t_1, and the random processes are again wide-sense stationary. Then the estimate of $\hat{x}(t_1)$ is based on the sample path y_τ, $\tau \in (-\infty, t_1)$ instead of y_τ, $\tau \in (-\infty,+\infty)$ and the filter is termed causal. The derivation of the causal solution is discussed in some detail

by any number of sources, see for example Wong [8], or the original work of Wiener [2]. The highlights of the solution are presented here for the scalar observation process case, as they are pertinent to subsequent causal filter discussions in this paper.

If the spectral density function of the observation process y_τ satisfies the Paley-Wiener condition,

$$\int_{-\infty}^{\infty} \frac{|\log S_y(\omega)|}{1+\omega^2} d\omega < \infty \tag{2.1.20}$$

then $S_y(\omega)$ may be factored into two multiplicative terms $S_y^+(\omega)$ and $S_y^-(\omega)$ which have the following properties:

$$S_y(\omega) = S_y^+(\omega) S_y^-(\omega) \tag{2.1.21}$$

$$\mathcal{F}^{-1}\{S_y^+(\omega)\} = 0 \quad \text{for } t < 0 \quad \text{(causal)} \tag{2.1.22}$$

$$\mathcal{F}^{-1}\{S_y^-(\omega)\} = 0 \quad \text{for } t > 0 \quad \text{(anti-causal)}. \tag{2.1.23}$$

The above conditions also imply

$$\mathcal{F}^{-1}\left\{\frac{1}{S_y^+(\omega)}\right\} = 0 \quad \text{for } t < 0 \quad \text{also}. \tag{2.1.24}$$

The fundamental result of Wiener states that if the Paley-Wiener condition is satisfied, as it is in most physically meaningful situations, then the optimal causal filter's transfer function, $H_{opt}^{\dagger}(\omega)$ is given by the expresssion

$$H_{opt}^{\dagger}(\omega) = \frac{1}{S_y^+(\omega)} \int_0^{\infty} e^{i\omega t} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{S_{xy}(\eta)}{S_y^-(\eta)} e^{i\eta t} d\eta \right] dt. \tag{2.1.25}$$

Defining

$$H_1(\omega) = \frac{S_{xy}(\omega)}{S_y^-(\omega)} , \tag{2.1.26}$$

it is easy to see that

$$H_{opt}^{\dagger}(\omega) = \frac{1}{S_y^+(\omega)} \int_0^{\infty} e^{i\omega t} h_1(t) dt. \tag{2.1.27}$$

Now define

$$h_2(t) = \begin{cases} h_1(t) & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t < 0 \end{cases} . \tag{2.1.28}$$

Then

$$H_{opt}^{\dagger}(\omega) = \frac{1}{S_y^+(\omega)} H_2(\omega), \tag{2.1.29}$$

which will be referred to henceforth as

$$H_{opt}^{\dagger}(\omega) = \frac{1}{S_y^+(\omega)} \left[\frac{S_{xy}(\omega)}{S_y^-(\omega)} \right]_+ . \tag{2.1.30}$$

A subscripted + (or -) sign denotes $H_2(\omega)$ ($\tilde{H}_2(\omega)$), which may be found by inverse Fourier transforming $H_1(\omega)$, truncating $h_1(t)$ by setting its anti-causal (causal) part to zero, and transforming the result to give $H_2(\omega)$ (or $\tilde{H}_2(\omega)$) as outlined above. This operation is frequently referred to as additive decomposition of $H_1(\omega)$ into causal and anti-causal components, since

$$H_1(\omega) = H_2(\omega) + \tilde{H}_2(\omega) \tag{2.1.31}$$

and $H_2(\omega)$ is causal while $\tilde{H}_2(\omega)$ is anti-causal. Again an explicit expression has been found for the optimal filter in terms of the spectral properties of

the random processes involved. For this causal problem, however, the spectral factorizations involved in computing the optimal transfer function are considerably more involved. This complexity is compounded when robust causal filtering problems are discussed later.

Recall that both of the special cases considered thus far are expressed in frequency domain terms. This type of analysis also leads to a convenient expression for the mean-square error under the system equations may be written in state-space form:

$$\dot{x}_t = Ax_t + B\xi_t \tag{2.1.32}$$

$$y_t = Cx_t + \theta_t. \tag{2.1.33}$$

ξ_t and θ_t are (n×1) and (m×1) wide-sense stationary, zero mean vector random processes such that

$$E\{\xi_t\ \xi_\tau^T\} = R_\xi(t-\tau), \tag{2.1.34}$$

$$E\{\theta_t\ \theta_\tau^T\} = R_\theta(t-\tau), \tag{2.1.35}$$

$$E\{\theta_t\ \xi_\tau^T\} = 0. \tag{2.1.36}$$

x_t is an (n×1) state vector, y_t is an (m×1) observation vector, and A and C are suitably dimensioned constant matrices.

Now if $\hat{X}(j\omega)$ is defined

$$\hat{X}(j\omega) \triangleq H(j\omega)Y(j\omega) = H(j\omega)CX(j\omega) + H(j\omega)\Theta(j\omega) \tag{2.1.37}$$

$\hat{X}(j\omega)$ forms a (not necessarily optimal) estimate of $X(j\omega)$, for which the mean-square error

$$\mathcal{E} = E\{(x_t-\hat{x}_t)^T Q(x_t-\hat{x}_t)\} \tag{2.1.38}$$

may be computed. Using the trace identity

$$\mathrm{tr}\{AB\} = \mathrm{tr}\{BA\} \tag{2.1.39}$$

and Parsenal's theorem, (2.1.38) becomes

$$\mathcal{E} = \frac{1}{2\pi}\,\mathrm{tr}\,E\{\,Q \int_{\mathbb{R}} X(+j\omega)X^T(-j\omega) - \hat{X}(+j\omega)X^T(-j\omega) - X(+j\omega)\hat{X}^T(-j\omega) + \hat{X}(+j\omega)\hat{X}^T(-j\omega)\,d\omega\}. \tag{2.1.40}$$

Now substitute equation (2.1.37) and note that for second-order θ_t and ξ_t, equation (2.1.36) implies

$$E\{\int_{\infty}^{\infty} \Theta(j\omega)\Xi^T(-j\omega)\,d\omega\} = 0. \tag{2.1.41}$$

Substituting into the error expression,

$$\mathcal{E} = \frac{1}{2\pi}\,\mathrm{tr}[E\{Q \int_{-\infty}^{\infty} [I-HC](j\omega I-A)^{-1}B\Xi(j\omega)\Xi^H(+j\omega)B^T(+j\omega I-A)^{-H}[I-HC]^H d\omega + Q\int_{-\infty}^{\infty} H\Theta(j\omega)\Theta^H(j\omega)H^H d\omega\}]. \tag{2.1.42}$$

After simplification,

$$\mathcal{E} = \frac{1}{2\pi}\,\mathrm{tr}[Q \int_{-\infty}^{\infty} [I-HC](j\omega I-A)^{-1}BS_\xi B^T(j\omega I-A)^{-H}[I-HC]^H d\omega + Q\int_{-\infty}^{\infty} HS_\varepsilon H^H d\omega]. \tag{2.1.43}$$

When all of the system parameters are known, the mean-square estimation error may be calculated directly using this result.

The two special cases presented thus far are tractable by virtue of their stationarity, and the consequent applicability of frequency domain analysis. By making different underlying assumptions on the system structure,

time domain analysis may be used to full advantage with techniques originated by Kalman and Bucy [1].

The details of Kalman filtering are largely irrelevant to the current discussion, yet a brief look at the assumed system structure and nature of the solution illustrate the close connection of Wiener filtering to the popular Kalman techniques.

Kalman presupposes a finite-dimensional state-space structure, which is in general time varying, but driven by white noise.

$$\dot{x}_t = A(t)x_t + \xi_t \quad , \qquad t \geq t_o \tag{2.1.44}$$

$$y_t = C(t)x_t + \theta_t \quad , \qquad t \geq t_o \tag{2.1.45}$$

ξ_t and θ_t are (n×1) and (m×1), respectively, vector zero mean white noise processes such that

$$E\{\xi_t \; \xi_\tau^T\} = S_\xi(t)\delta(t-\tau), \tag{2.1.46}$$

$$E\{\theta_t \; \theta_\tau^T\} = S_\theta(t)\delta(t-\tau) \tag{2.1.47}$$

and

$$E\{\theta_t \; \xi_\tau^T\} = 0. \tag{2.1.48}$$

S_ξ and S_θ are non-negative definite time-varying constant matrices in accordance with the white noise requirement. x_t is the n×1 state vector and y_t the m×1 observation vector. A(t) and C(t) are suitably dimensional matrices of time functions. This type of structure is frequently encountered in many applications. Using state augmentation methods (see Kwakernaak and Sivan [9]), it may accommodate linear systems driven by a noise process which is not white, but can be represented by the output of a finite-dimensional linear system driven by white noise.

Kalman's solution consists of a dynamic system of the form

$$\dot{x}_t = A(t)x_t + L(t)[y_t - C(t)x(t)], \quad t \geq t_o \tag{2.1.49}$$

$$\dot{x}(0) = 0 \tag{2.1.50}$$

where L(t) is given by

$$L(t) = \Sigma(t)C^T(t)(S_\theta(t))^{-1} \tag{2.1.51}$$

for $t \geq t_o$, and $\Sigma(t)$ is the solution to the matrix Riccati equation

$$\dot{\Sigma}(t) = A(t)\ \Sigma(t) + \Sigma(t)A^T(t) + S_\xi(t) - \Sigma(t)C^T(t)S_\theta^{-1}(t)C(t)\Sigma(t) \tag{2.1.52}$$

with boundary condition

$$\Sigma(t_o) = S_\xi(t_o). \tag{2.1.53}$$

It is interesting to note the connection between the time-invariant causal Wiener filtering scheme presented for stationary second-order processes and the Kalman-Bucy filter. Although the causal Wiener filtering arguments were developed on the assumption that the noise processes are of second-order, slight modifications to the same arguments show that the same results also apply to stationary white noise processes. Consequently, there is a class of problems to which both methods may be used. These problems are time-invariant with stationary process and observation noises (required for Wiener solution), and linear and finite dimensional with white process and observation noises (required for Kalman solution). For these problems, the steady-state ($t >> t_o$) Kalman solution approaches the causal Wiener solution (recall that for Wiener filtering $t_o = -\infty$, so until $t_o - t \doteq -\infty$ the solutions will differ).

The entire preceding discussion of optimal filtering has been presented in order to review well-known techniques of filtering an observation process to obtain an estimate of a state process in the presence of random additive noise in both state and observation assuming:

i) A reasonable model of the system dynamics is available.

ii) The relationship between the state process of the system and the observed process is known.

iii) The noise processes are second-order and have known second-order statistics.

Various additional assumptions on the nature of the problem lead to different methods of analytical solution. Also, the state-space description which provides a valuable internal description of a system is easily accommodated in each of the procedures mentioned. Frequently, the state and observation dynamics are well understood, but assuming that the second-order statistics of the noise processes involved are known exactly may be somewhat naive from a design standpoint. Consequently, a superior filter might be sought which provides acceptable error performance over a range of anticipated noise process statistics.

2.2. One-Dimensional Robust Wiener Filtering

Most of the work published to date on optimal robust filter design is concerned with estimating a scalar state process from a scalar observation. Some of the significant results are presented in this section.

The one-dimensional system model has the form

$$y_t = (g_c(t) * x(t)) + \theta_t \tag{2.2.1}$$

$$x_t = g_x(t) * \xi_t, \tag{2.2.2}$$

where an asterisk indicates the convolution operation

$$g_x(t) * \xi(t) = \int_{-\infty}^{\infty} g_x(\sigma)\xi(t-\sigma)d\sigma, \tag{2.2.3}$$

and all quantities are scalars. The input, $\xi(t)$ and observation, $\theta(t)$ scalar random processes are second order, and $x(t)$ is the process to be estimated. The primary purpose for considering such a model is to allow the estimation of an internal system quantity. For instance, the problem of estimating one element of the state vector of a finite dimensional system as in equations (2.1.32)-(2.1.36) fits this formulation. Therefore x_t will henceforth be referred to as the state process, though it may or may not correspond to all or part of the system state vector.

Any systems of this form are suitable for Wiener filtering since

$$S_{xy}(\omega) = |G_x(j\omega)|^2 S_\xi(\omega)[G_c(j\omega)]^H \tag{2.2.4}$$

$$S_y(\omega) = |G_c(j\omega)G_x(j\omega)|^2 S_\xi(\omega) + S_\theta(\omega) \tag{2.2.5}$$

are expressed in terms of the noise process PSD's, $S_\xi(\omega)$ and $S_\theta(\omega)$. The superscript denotes the Hermitian transpose (complex conjugation in the scalar case here), and

$$|G(j\omega)|^2 = G^H(j\omega)G(j\omega). \tag{2.2.6}$$

A useful, albeit conservative definition of the optimal robust filter is the minimax estimator. The minimax filter minimizes over all allowable filter structures the worst case mean-square error. Stated more formally, the estimator has a transfer function given by

$$H^R(j\omega) = \arg \min_{H(\omega) \in \mathcal{H}} \{ \max_{(S_\xi, S_\theta) \in \mathcal{X} \times \mathcal{N}} \mathcal{E}(H, S_\xi, S_\theta) \}. \quad (2.2.7)$$

$\mathcal{H}$ indicates the set of allowable filter structures. In light of the earlier discussion, two useful classes for $\mathcal{H}$ are the sets of causal transfer functions and the set of non-causal transfer functions. $\mathcal{X}$ and $\mathcal{N}$, respectively comprise the sets of power spectra to which $S_\xi(\omega)$ and $S_\xi(\omega)$ are expected to belong.

A central issue in such a robust design procedure is the establishmetn of appropriate spectral classes $\mathcal{X}$ and $\mathcal{N}$ for the particular problem at hand. Though many possible classes are conceivable, a few ways of specifying spectral classes have come into common use in robust filtering for various analytical and physical reasons.

Consider designing a filter for a one-dimensional plant that is to be produced on an assembly line. Suppose that the sensor noise alone constitutes the entire observation noise process, and that each sensor may be tested before assembly, and the failed units discarded. A reasonable test of the sensors might measure the noise power at the sensor output and require that the total noise power not exceed a predetermined maximum. To ensure acceptable frequency response performance of the units, an upper bound on the sensor noise spectrum might be placed at each frequency. Such a physical application motivates one model for defining spectral classes known as the Spectral-Band model [4]. More precisely, a spectral class $\mathcal{N}$ may be identified

$$\mathcal{N} = \{ S_\theta(\omega) \,|\, S_L(\omega) \le S_\theta(\omega) \le S_u(\omega), \;\; \frac{1}{2\pi} \int_{-\infty}^{\infty} S_\theta(\omega) d\omega = P_\theta \}. \quad (2.2.8)$$

The first condition establishes confidence limits between which $S_\theta(\omega)$ is expected to lie for all ω. The second condition specifies the overall noise power. Obviously this restriction is required for a minimax procedure to provide solutions other than trivial results like: the greatest noise power is least favorable for filtering, therefore the least favorable spectrum is $S_\theta(\omega) = S_u(\omega)$.

Now consider an entirely different design setting - a suitable spectral class for the state noise is to be chosen when the state process consists of a signal received through a communication channel. The state process noise might be expected to consist of a background band-limited white Gaussian noise and additional noise from interfering channels and the like of unknown spectral form. In this situation a mixture model more accurately fits the application

$$\mathcal{X} = \{S_\xi(\omega) \,|\, (1-\varepsilon)S_N(\omega) + \varepsilon S_C(\omega), \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} S_N(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_C(\omega)\,d\omega = P_\xi\}. \quad (2.2.9)$$

ε is a parameter which can be chosen in accordance with an estimate of the ratio of noise powers in the known nominal spectrum $S_N(\omega)$ and unknown contamination spectrum $S_C(\omega)$. Again the power constraint is included to effect meaningful solutions.

Another type of spectral class based on a nominal spectrum which does not require a lower spectral bound is the total-variation model. Good for significant analytical mileage (see Poor [5]), it is stated

$$\mathcal{X} = \{S_\xi(\omega) \,|\, \frac{1}{2\pi}\int |S_N(\omega) - S_\xi(\omega)|\,d\omega \le \beta_\xi, \quad \frac{1}{2\pi}\int S_\xi(\omega)\,d\omega = P_\xi\}. \quad (2.2.10)$$

The task of identifying appropriate spectral classes for a design is by no means trivial and greatly determines the relevance of the analysis. Furthermore, for a particular situation, none of these analytically expedient models may suffice. Yet the aim of this paper is not to discuss all of the design possibilities, but to develop techniques for extending one-dimensional robust filtering problems to the multivariable setting. From this point onward, discussion and illustrative examples will center around the spectral-band model, partially due to its physical significance, but mostly on account of its analytical convenience. The relationship between the different spectral class models and their corresponding solutions is treated in more detail by Kassam [10].

Now that the minimax problem formulation has been completed, a method of solution is required. The error expression for one-dimensional time invariant problems of the form (2.2.1)-(2.2.2) is given by

$$\mathcal{E} = \frac{1}{2\pi} \int_{\infty}^{\infty} |1-H(j\omega)G_c(j\omega)|^2 |G_x(j\omega)|^2 S_\xi(\omega) + |H(j\omega)|^2 S_\theta(\omega)\, d\omega. \tag{2.2.11}$$

This follows directly from equation (2.1.43). Since $S_x(\omega)$ is a function of $S_\xi(\omega)$, the error is a function of H, S_ξ, and S_θ. The minimax problem may be written

$$\min_{H\in\mathcal{H}} \{ \max_{(S_\xi, S_\theta)\in \mathcal{X}\times\mathcal{N}} \mathcal{E}(H, S_\xi, S_\theta)\}. \tag{2.2.12}$$

This type of problem requires that both the maximization and minimization be performed simultaneously. The following well known theorem of Barbu and Precupanu [11] states that saddlepoint solutions to the minimax problem also are saddlepoint solutions to the associated maximin problem

$$\max_{(S_\xi,S_\theta)\in\mathcal{X}\times\eta} \{\min_{H\in\mathcal{H}} \mathcal{E}(H,S_\xi,S_\theta)\} \tag{2.2.13}$$

for convex spectral classes $\mathcal{X}$ and η.

<u>Theorem 2.1</u>: There exists a triplet $(H^R,S_\xi^R,S_\theta^R)\in\mathcal{H}\times\mathcal{X}\times\eta$ satisfying the saddlepoint condition

$$\mathcal{E}(H^R,S_\xi,S_\theta) \le \mathcal{E}(H^R,S_\xi^R,S_\theta^R) \le \mathcal{E}(H,S_\xi^R,S_\theta^R) \quad \text{for all } H\in\mathcal{H},\ S_\xi\in\mathcal{X},\ S_\theta\in\eta \tag{2.2.14}$$

if and only if the solutions of the minimax and maximin problems are equal. The triplet (H^R,S_ξ^R,S_θ^R) is the solution to both problems.

<u>Proof</u>: See [11].

This theorem greatly simplifies the robust filtering problem in that if a solution to the maximin problem may be found which satisfies the saddlepoint condition, it also forms a solution to the minimax problem of interest. The triplet solution consists of the optimal filter transfer function, and a pair of spectra S_ξ^R and S_θ^R which may be called least favorable for Wiener filtering.

Recall from the first section of this chapter that for time-invariant Wiener filtering in both the causal and non-causal cases, the optimal filter transfer function is a solution to the bracketed quantity of the maximin problem and may be written in terms of S_ξ and S_θ. This property of the optimal filter effectively separates the maximin problem into two separate extremal problems and gives an immediate solution to the inner minimization. The maximin problem then is reduced to

$$\max_{(S_\xi,S_\theta)\in\mathcal{X}\times\eta} \{\mathcal{E}(S_\xi,S_\theta)\} \tag{2.2.15}$$

with the appropriate expression for $H_{opt}(j\omega)$ inserted in the error expression. This technique is useful in solving many robust filtering problems. Using this type of reasoning, Kassam and Lim have found non-causal solutions for problems with spectral band and mixture model spectral uncertainties [4], and Poor has done the same for total variation models [5]. The causal problems do not provide a general explicit error function $\mathcal{E}(S_\xi, S_\theta)$ for use in equation (2.2.15), so solutions to robust causal problems may only be found in specific instances for which such an error expression exists. Poor has demonstrated solutions when either S_θ or $|G_y|^2 S_\xi$ is known and wide-sense Markov [5] or white (Poor and Looze, [6]), as well as a number of other specific problems [5,12]. Kassam and Lim's non-causal spectral-band result [4] is of fundamental significance to this paper and is stated almost word-for-word in the Appendix. The robust pair of spectra found by their method have intuitively satisfying properties which might be expected of least favorable spectra for filtering. Particularly, note that in the regions of the frequency spectrum where the upper or lower bounds of the uncertainty model are not encountered, the robust spectrum is proportional to one of the bounds on the other process (in cases a and c).

To illustrate this solution, consider the following example:

Example 2.1:

$$y_t = x_t + \theta_t \tag{2.2.16}$$

$$x_t = g_x(t) * \xi_t \tag{2.2.17}$$

with

$$g_x(t) = \mathcal{F}^{-1}\left\{\frac{\sqrt{8}}{j\omega - 4}\right\}. \tag{2.2.18}$$

Then

$$S_x(\omega) = \left|\frac{\sqrt{8}}{j\omega-4}\right|^2 S_\xi(\omega). \tag{2.2.19}$$

Assume that the process noise ξ is white and of unit spectral height ($S_\xi(\omega) = 1,\ \forall\omega$), and the observation process noise θ is known to lie within a spectral band model

$$S_L = \frac{1}{\omega^2+1} \le S_\theta(\omega) \le \frac{3}{\omega^2+1} = S_u \tag{2.2.20}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_\theta(\omega)d\omega = P_\theta. \tag{2.2.21}$$

By case A of Theorem A.1 (in the Appendix), the least favorable observation noise spectrum is then

$$S_x^R(\omega) = \frac{8}{\omega^2+16} \tag{2.2.22}$$

$$S_\theta^R(\omega) = \left\{\begin{array}{cc} N_L \quad , & (1/k_n)S_x < N_L(\omega) \\ (\frac{1}{k_n})S_x(\omega), & N_L(\omega) \le (1/k_n)S_x \le N_u(\omega) \\ N_u \quad , & (1/k_n)S_x > N_u(\omega) \end{array}\right\}, \tag{2.2.23}$$

where k_n is chosen to satisfy the power constraint of equation (2.2.21). The robust spectrum if $P_\theta = .847$, for instance, is shown in Figure 2.1 of the following page as a heavy solid line (for which $k_n = 1$). The robust non-causal filter is given by

$$H^R(j\omega) = \frac{S_x^R(\omega)}{S_x^R(\omega) + S_\theta^R(\omega)}. \tag{2.2.24}$$

Calculating values when $k_n = 1$,

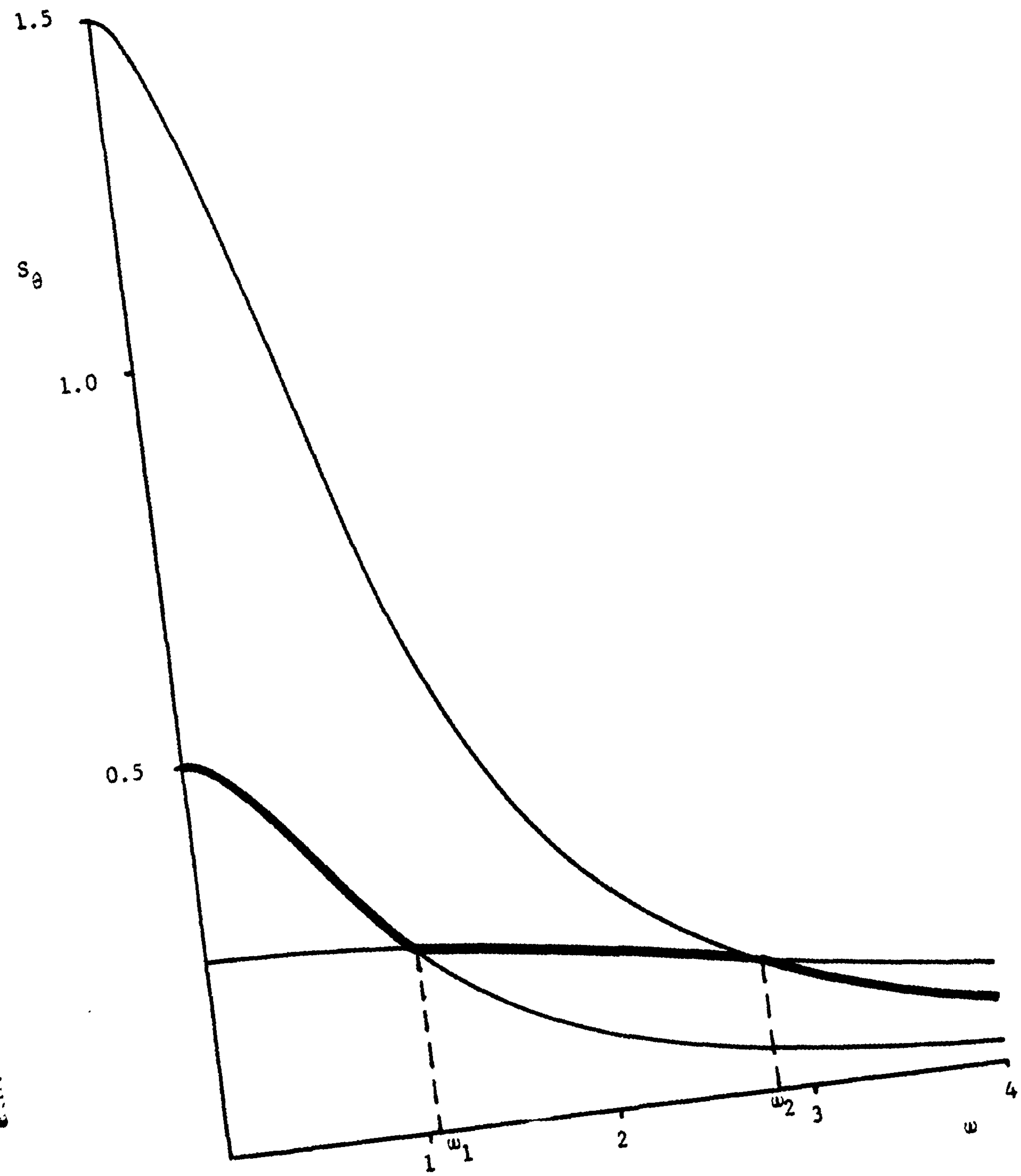

Figure 2.1 The Least Favorable Spectrum

$$H^R(j\omega) = \begin{cases} \frac{8}{3}\left[\frac{\omega^2+1}{3\omega^2+8}\right], & |\omega| < 1.07 \\ \frac{1}{2}, & 1.07 \leq |\omega| \leq 2.83 \\ 8\left[\frac{\omega^2+1}{11\omega^2+56}\right], & |\omega| > 2.83 \end{cases} \qquad (2.2.25)$$

and the solution is complete.

Summarily, the procedure for solving minimax robust estimation problems is conceptually routine once the relationship between the minimax and maximin problems is recognized. Useful results may be obtained for non-causal problems under several different spectral uncertainty models. Similar extensions to causal problems do not follow directly from the maximin problem, for lack of a closed-form expression for the mean square filtering error in terms of the process and observation noise spectra. Consequently, results may be obtained only for special cases in which such error expressions can be found. Specific results have been presented, for a non-causal spectral band problem for purposes of the subsequent multiple-state discussion.

3. MULTIPLE-STATE ROBUST ESTIMATION

This section is concerned with extending one-dimensional robust filtering techniques to higher dimensional problems. A particular multiple-state model is presented and justified as reasonable for a first investigation into multivariable robust filtering. Two multiple-state minimax formulations are proposed for finding multiple-state estimators. Equivalence of the solutions to the different formulations is established for systems which fit the multiple-state model without causality constraints on the filter structure and for which the noise processes are specified by spectral-band models, and a design procedure follows directly. For other types of non-causal problems a more general, though slightly less computationally expedient method is devised. Analogous causal results (or lack thereof) are discussed briefly; an investigation of the error bounds associated with the various methods concludes the section.

3.1. The Multiple-State Model

The system model to be used throughout this section may be represented in the time domain by the following equations:

$$x(t) = g_x(t) * \xi(t) \tag{3.1.1}$$

and

$$y(t) = (g_c(t) * x(t)) + \theta(t). \tag{3.1.2}$$

Again $*$ denotes time-domain convolution. $\xi(t)$, $\theta(t)$ and $y(t)$ are scalar processes. $\theta(t)$ and $\xi(t)$ are either second-order or white. $x(t)$ is an n-dimensional "state-process" vector, and $g_x(t)$ and $g_c(t)$ are compatibly dimensioned impulse response functions. Additionally $\xi(t)$ and $\theta(t)$ are

zero mean and uncorrelated so

$$E\{\xi_t\,\theta_\tau\} = 0 \tag{3.1.3}$$

$$\mathfrak{F}[E\{\xi_t\,\xi_\tau\} = S_\xi(\omega) \tag{3.1.4}$$

and

$$\mathfrak{F}[E\{\theta_t\,\theta_\tau\} = S_\theta(\omega)\ , \tag{3.1.5}$$

for all $t, \tau \in \mathbb{R}$.

A more universally applicable model would allow for vector noise processes and correlation between process and observation noises. There are several drawbacks with such a general model. The calculations become rather cumbersome when vector noise processes are considered (particularly the spectral decompositions involved in causal problems), and could unnecessarily obfuscate the character of the problem. Before vector noise processes may be studied, more elaborate matrix spectral uncertainty classes must be established, while scalar noise processes may be studied using the spectral classes already established for one-dimensional robust filtering. The multiple-state model is therefore a most attractive one for the investigation.

The multiple-state model nonetheless includes a large variety of problems. Any systems represented in state-space form (see eqns.(2.1.32-2.1.36)) fit this model, and the impulse response matrices may be identified

$$g_x(t) = \mathfrak{F}^{-1}\{G_x(j\omega)\} = \mathfrak{F}^{-1}\{(j\omega I-A)^{-1}B\}, \tag{3.1.6}$$

$$g_c(t) = \mathfrak{F}^{-1}\{G_c(j\omega)\} = \mathfrak{F}^{-1}\{C\} = C\delta(t), \tag{3.1.7}$$

and

$$g_y(t) = g_x(t) * g_c(t). \tag{3.1.8}$$

Such representations frequently apply to practical problems, the double integrator plant example of Section 4 for example is in state-space form.

3.2. The State-by-State and Least-Squares Formulations

Now that the multiple-state system to be considered has been established, the minimax robust filter problem may be stated. Since the quantity $(x-\hat{x})$ is a vector, two definitions of the most robust filter may be presented. The robust filter

$$H^R(j\omega) = \begin{bmatrix} H_1^R(j\omega) \\ H_2^R(j\omega) \\ \cdot \\ \cdot \\ \cdot \\ H_n^R(j\omega) \end{bmatrix} \tag{3.2.1}$$

may be designed by selecting $H_1^R(j\omega)$ to minimize the worst case error in x_1, $H_2^R(j\omega)$ to minimize the worst case error in x_2, and so on. A filter which satisfies this requirement will be called <u>state-by-state</u> robust. Alternatively, a robust filter may be defined as one which minimizes some vector norm of $(x-\hat{x})$. The most general mean-square norm is

$$\mathcal{J} = E\{(x-\hat{x})^T Q(x-\hat{x})\}. \tag{3.2.2}$$

For the purposes of this paper, attention will be restricted to problems where

$$Q = I. \tag{3.2.3}$$

Most of the results obtained may be applied with minor modification for all positive semi-definite weighting matrices Q, but the notation becomes inconvenient and adds little to the results. This case corresponds to minimizing the sum of the squares of the state errors, and will henceforth be referred to as a <u>least-squares</u> robust problem. It is a special case of the more general norm-wise robust formulation.

In either case, the robust filter is given by equation (3.2.1). A state-by-state robust filter satisfies

$$\min_{H_i(j\omega) \in \mathcal{K}} \left\{ \max_{(S_\xi, S_\theta) \in \mathcal{X} x \mathcal{N}} E\,[(x_i-\hat{x}_i)^2] \right\}. \tag{3.2.4}$$

Calculation of this robust filter involves solving n one-dimensional minimax problems by methods similar to those of Section 2. The least-squares robust filter problem may be stated

$$\min_{H(j\omega) \in \mathcal{K}^n} \left\{ \max_{(S_\xi, S_\theta) \in \mathcal{X} x \mathcal{N}} E\,[(x-\hat{x})^T (x-\hat{x})] \right\} \tag{3.2.5}$$

This formulation is quite attractive since it involves a single minimax problem and reflects the quadratic cost criterion used extensively in control theory literature. However, such a formulation cannot be solved

using one-dimensional techniques directly. The relationship between state-by-state and least-squares formulations and their solutions is central to an adequate robust design procedure.

3.3. Solution of Non-Causal Problems

This section is divided into two subsections. In subsection 3.3.1 a general approach is established for solution of the state-by-state formulation. Subsection 3.3.2 deals specifically with the application of this method to spectral-band problems and the implications of the result.

3.3.1. General State-by-State Solution

The one-dimensional problems discussed in Section 2 each resulted in maximin problems of the form

$$\max_{(\tilde{S}_x,\tilde{S}_\theta)\in\tilde{\mathcal{X}}\times\tilde{\mathcal{N}}}\left\{\min_{\tilde{H}(j\omega)\in\mathcal{H}}\frac{1}{2\pi}\int_{-\infty}^{\infty}|1-\tilde{H}(j\omega)|^2\tilde{S}_x+|\tilde{H}(j\omega)|^2\tilde{S}_\theta d\omega\right\}. \tag{3.3.1.1}$$

Solution of the non-causal minimization gives

$$\max_{(\tilde{S}_x,\tilde{S}_\theta)\in\tilde{\mathcal{X}},\tilde{\mathcal{N}}}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}|1-\tilde{H}_{opt}(j\omega)|^2\tilde{S}_x+|\tilde{H}_{opt}(j\omega)|^2\tilde{S}_\theta d\omega\right\}, \tag{3.3.1.2}$$

where

$$\tilde{H}_{opt}(j\omega) = \frac{\tilde{S}_x}{\tilde{S}_x+\tilde{S}_\theta} \tag{3.3.1.3}$$

Saddlepoint solutions for this problem are already established for many

different spectral class models. The solution for spectral-band models is presented in the Appendix in some detail.

The state-by-state formulation produces n maximin problems of the form

$$\max_{(S_\xi, S_\vartheta) \in \mathcal{Z}\times\mathcal{N}} \left\{ \min_{H_i(j\omega) \in \mathcal{H}} \frac{1}{2\pi} \int_{-\infty}^{\infty} |G_{x_i} - H_i G_y|^2 S_\xi + |H_i|^2 S_\vartheta \, d\omega \right\}, \qquad (3.3.1.4)$$

with G_{x_i} the transfer function between the input ξ_t and the state x_i, and G_y the transfer function from ξ_t to y. Factoring G_{x_i} from the first term of the integrand,

$$\max_{(S_\xi, S_\vartheta) \in \mathcal{Z}\times\mathcal{N}} \left\{ \min_{H_i(j\omega) \in \mathcal{H}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|1 - \frac{G_y}{G_{x_i}} H_i\right|^2 |G_{x_i}|^2 S_\xi + \left|\frac{G_y}{G_{x_i}} H_i\right|^2 \frac{|G_{x_i}|^2}{|G_y|^2} S_\vartheta \, d\omega \right\} \qquad (3.3.1.5)$$

This is exactly the form of equation (3.3.1) if the following quantities are identified:

$$\tilde{H}_i(j\omega) \triangleq \frac{G_y}{G_{x_i}} H_i(j\omega), \qquad (3.3.1.6)$$

$$\tilde{S}_{x_i}(j\omega) \triangleq |G_{x_i}|^2 S_\xi(\omega), \qquad (3.3.1.7)$$

$$\tilde{S}_{\vartheta_i}(j\omega) \triangleq \frac{|G_{x_i}|^2}{|G_y|^2} S_\vartheta(\omega), \qquad (3.3.1.8)$$

$$\tilde{\mathcal{Z}} = \left\{ \tilde{S}_x : \tilde{S}_{x_i}(\omega) = |G_{x_i}|^2 S_\xi, \quad S_\xi \in \mathcal{Z} \right\} \qquad (3.3.1.9)$$

and

$$\tilde{\mathcal{N}} = \left\{ \tilde{S}_\theta : \tilde{S}_{\theta_i}(\omega) = \frac{|G_{x_i}|^2}{|G_y|^2} S_\theta,\ S_\theta \in \mathcal{N} \right\}, \tag{3.3.1.10}$$

where $\mathcal{X}$ and $\mathcal{N}$ are the spectral uncertainty classes of the original problem. Solving the "tilde" one-dimensional problem by one-dimensional methods, such as using Theorem A.1 (see Appendix) for spectral band problems, gives a solution triplet $[\tilde{H}^R, \tilde{S}^R_{x_i}, \tilde{S}^R_{\theta_i}]$. The corresponding solution in the original coordinates may be found using equations (3.3.1.6)-(3.3.1.10) to give the maximin (and minimax) solution to the original problem, $[H^R_i, S^R_{x_i}, S^R_{\theta_i}]$.

3.3.2. Solution for Spectral-Band Uncertainty Models

Consider non-causal spectral-band problems, where $\mathcal{X}$ and $\mathcal{N}$ are spectral uncertainty classes of the form of equation (2.2.8). $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{N}}$ are then also spectral-band classes with

$$\tilde{S}_{L_i} = |G_{x_i}|^2 S_L, \tag{3.3.2.1}$$

$$\tilde{S}_{U_i} = |G_{x_i}|^2 S_U, \tag{3.3.2.2}$$

$$\tilde{N}_{L_i} = \frac{|G_{x_i}|^2}{|G_y|^2} N_L, \tag{3.3.2.3}$$

and

$$\tilde{N}_{U_i} = \frac{|G_{x_i}|^2}{|G_y|^2} N_U. \tag{3.3.2.4}$$

The solution to the one-dimensional tilde problem (3.3.1.2) then is given by Kassam and Lim's theorem (Theorem A.1 of the Appendix). Each of the three

possible cases will be considered here.

Case A. Consider non-causal spectral-band problems for which $K_s \leq K_n$ gives a least favorable spectral pair for the tilde problem of the form

$$\tilde{S}^R_{x_i}(\omega) = \begin{cases} \tilde{S}_{L_i}, & \omega \in \bar{\alpha}_i\,(K_{s_i}) \\ K_{s_i}\tilde{N}_{L_i}, & \omega \in \alpha_{1_i}(K_{s_i}) \\ \tilde{S}_{U_i}, & \omega \in \alpha_{2_i}\,(K_{s_i}) \end{cases} \tag{3.3.2.5}$$

$$\tilde{S}^R_{\theta_i}(\omega) = \begin{cases} \tilde{N}_{L_i}, & \omega \in \bar{\beta}_i\,(K_{n_i}) \\ (1/K_{n_i})\tilde{S}_{L_i}, & \omega \in \beta_{1_i}(K_{n_i}) \\ \tilde{N}_{U_i}, & \omega \in \beta_{2_i}(K_{n_i}) \end{cases} \tag{3.3.2.6}$$

Using equations 3.3.7 - 3.3.10

$$|G_{x_i}|^2 S^R_{\xi_i} = \tilde{S}^R_x = \begin{cases} |G_{x_i}|^2 S_L, & \omega \in \bar{\alpha}_i(K_{s_i}) \\ K_{s_i}\dfrac{|G_{x_i}|^2}{|G_y|^2}\,N_L, & \omega \in \alpha_{1_i}(K_{s_i}) \\ |G_{x_i}|^2\,S_U, & \omega \in \alpha_{2_i}(K_{s_i}) \end{cases} \tag{3.3.2.7}$$

and

$$\frac{|G_{x_i}|^2}{|G_y|^2} S^R_{\theta_i} = \tilde{S}^R_\theta = \begin{cases} \dfrac{|G_{x_i}|^2}{|G_y|^2}\,N_L, & \omega \in \bar{\beta}_i(K_{n_i}) \\ (1/K_{n_i})|G_{x_i}|^2\,S_L, & \omega \in \beta_{1_i}(K_{n_i}) \\ \dfrac{|G_{x_i}|^2}{|G_y|^2}N_U, & \omega \in \beta_{2_i}(K_{n_i}) \end{cases} \tag{3.3.2.8}$$

The state-by-state solution triplet then consists of the least favorable spectra

$$S^R_{\xi_i}(\omega) = \begin{cases} S_L, & S_L > \dfrac{K_S}{|G_y|^2} N_L \\ \dfrac{K_S}{|G_y|^2} N_L, & S_L \le \dfrac{K_S}{|G_y|^2} N_L \le S_U \\ S_U & S_U < \dfrac{K_S}{|G_y|^2} N_L \end{cases} \tag{3.3.2.9}$$

and

$$S^R_{\theta_i}(\omega) = \begin{cases} N_L, & N_L > \dfrac{|G_y|^2}{K_n} S_L \\ \dfrac{|G_y|^2}{K_n} S_L, & N_L \le \dfrac{|G_y|^2}{K_n} S_L \le N_U \\ N_U, & N_U < \dfrac{|G_y|^2}{K_n} S_L \end{cases} \tag{3.3.2.10}$$

(K_S is the same for each state by the power constraint)and the robust filter

$$H^R_i(j\omega) = \frac{G_{x_i} G^*_y S^R_{\xi_i}}{|G_y|^2 S^R_{\xi_i} + S^R_\theta} \tag{3.3.2.11}$$

which follows from equation(3.3.1.3). Note that the state index i does not appear in the expressions for the robust spectra.

<u>Case B</u>. For problems with tilde solutions which fall under "case b" of Theorem A.1, similar arguments can be made. The resulting solutions are

$$S^R_{\xi_i}(\omega) = \begin{cases} \dfrac{K_i}{|G_y|^2} N_L + \dfrac{1}{|G_{x_i}|^2} \tilde{S}_{e_i}, & \omega \in \alpha_{1_i}(K) \\[2ex] S_U, & \omega \in \alpha_{2_i}(K) \\[2ex] S_L + \dfrac{1}{|G_{x_i}|^2} \tilde{S}_{e_i}, & \omega \in \beta_{1_i}(K) \\[2ex] S_L, & \omega \in \beta_{2_i}(K) \end{cases} \tag{3.3.2.12}$$

and

$$S^R_{\theta_i}(\omega) = \begin{cases} \dfrac{|G_y|^2}{K_i} S_L + \dfrac{|G_y|^2}{|G_{x_i}|^2} \tilde{N}_e, & \omega \in \beta_{1_i}(K) \\[2ex] N_U, & \omega \in \beta_{2_i}(K) \\[2ex] N_L + \dfrac{|G_y|^2}{|G_{x_i}|^2} \tilde{N}_e, & \omega \in \alpha_{1_i}(K) \\[2ex] N_L, & \omega \in \alpha_{2_i}(K) \end{cases} \tag{3.3.2.13}$$

with the filter given by equation (3.3.2.11). From the theorem, it is required that

$$\tilde{S}_{e_i}(\omega) = K_i \tilde{N}_{e_i}(\omega). \tag{3.3.2.14}$$

If the functions $S_{e_i}(\omega)$ and $N_{e_i}(\omega)$ are defined

$$S_{e_i}(\omega) = \frac{1}{|G_{x_i}|^2} \tilde{S}_e(\omega) \qquad (3.3.2.15)$$

and

$$N_{e_i}(\omega) = \frac{1}{|G_{x_i}|^2} \tilde{N}_e(\omega) , \qquad (3.3.2.16)$$

then the least favorable spectra may be written

$$S^R_{\xi_i}(\omega) = \begin{cases} \dfrac{K}{|G_y|^2} N_L + S_e , & \omega \in \alpha_1(K) \\ S_U , & \omega \in \alpha_2(K) \\ S_L + S_e , & \omega \in \beta_1(K) \\ S_L , & \omega \in \beta_2(K) \end{cases} \qquad (3.3.2.17)$$

and

$$S^R_{e_i}(\omega) = \begin{cases} \dfrac{|G_y|^2}{K} S_L + |G_y|^2 N_e , & \omega \in \beta_1(K) \\ N_U , & \omega \in \beta_2(K) \\ N_L + |G_y|^2 N_e , & \omega \in \alpha_1(K) \\ N_L , & \omega \in \alpha_2(K) \end{cases} \qquad (3.3.2.18)$$

Again the regions α_1, α_2, β_1 and β_2 remain invariant under the transformation and K is the same for each state by the power constraint, so once again the least favorable spectra do not depend on the state.

<u>Case C</u>. For these problems the final solutions for the least favorable spectra are

$$S^R_{\xi_i}(\omega) = \begin{cases} S_L , & \omega \in b_2(\ell_s) \\ \dfrac{\ell_s}{|G_y|^2} N_U , & \omega \in b_1(\ell_s) \\ S_U, & \omega \in \overline{b}(\ell_s) \end{cases} \tag{3.3.2.19}$$

and

$$S^R_{\theta_i}(\omega) = \begin{cases} N_L , & \omega \in a_2(\ell_n) \\ \dfrac{|G_y|^2}{\ell_n} S_U , & \omega \in a_1(\ell_n) \\ N_U , & \omega \in \overline{a}(\ell_n) \end{cases} \tag{3.3.2.20}$$

The point to be made is that for each of the three possible saddle-point solutions consisting of a least favorable spectral pair given by (3.3.2.) and (3.3.2.10), (3.3.2.17) and (3.3.2.18) or (3.3.2.19) and (3.3.2.20) and robust filter structure given by (3.3.2.11), the least favorable spectra <u>do not depend on i at all</u>. Therefore, the least favorable spectra pair $[S^R_{\xi_i},$ $S^R_{\theta_i}]$ is the same for all i, and may be denoted simply $[S^R_\xi, S^R_\theta]$.

This observation leads directly to a theorem concerning the relationship of the state-by-state and least-squares solutions of this particular problem.

<u>Theorem 3.1</u>

For multiple-state systems of the type described in Section 3.1, with $\mathcal{X}$ and $\mathcal{N}$ given by spectral band uncertainty models, and $\mathcal{H}$ the class of non-causal linear filters, the solutions to the state-by-state and the

least-squares problems are equal.

Proof: Define J_{SS} as the sum of the maximin errors due to each state in the state-by-state approach

$$J_{SS} \triangleq \sum_{i=1}^{n} \max_{(S_{\xi_i},\, S_{\theta_i}) \in \mathcal{X} \times \mathcal{N}} \left\{ \min_{H_i \in \mathcal{H}} E\,\{(x_i - \hat{x}_i)^2\} \right\}. \qquad (3.3.2.21)$$

By the arguments presented above, maximization occurs for $[S_{\xi_i}, S_{\theta_i}] = [S_\xi^R, S_\theta^R]$ for each state independent of i, therefore

$$J_{SS} = \sum_{i=1}^{n} \max_{(S_\xi, S_\theta) \in \mathcal{X} \times \mathcal{N}} \left\{ \min_{H_i \in \mathcal{H}} E\,\{(x_i - \hat{x}_i)^2\} \right\} \qquad (3.3.2.22)$$

and

$$J_{SS} = \max_{(S_\xi, S_\theta) \in \mathcal{X} \times \mathcal{N}} \sum_{i=1}^{n} \min_{H_i \in \mathcal{H}} E\,\{(x_i - \hat{x}_i)^2\}. \qquad (3.3.2.23)$$

Furthermore

$$J_{SS} = \max_{(S_\xi, S_\theta) \in \mathcal{X} \times \mathcal{N}} \left\{ \min_{H \in \mathcal{H}^n} E\,\{ \sum_{i=1}^{n} (x_i - \hat{x}_i)^2 \} \right\}, \qquad (3.3.2.24)$$

which is precisely the least-squares formulation. It follows immedeately that whenever the least favorable spectra are the same for each state's maximin problem, the state-by-state and least-squares solutions are identical. This completes the Proof.

A convenient design procedure follows immedeately from these results. For multiple-state non-causal, spectral-band problems, the least favorable spectral pair may be found by solution of any one of the state-by-state formulation's maximin problems using Theorem A.1. The

resulting spectral pair, used in equation (3.3.2.11) with the appropriate transfer functions gives directly the robust filter's transfer function for each state x_i.

It should be mentioned that all of this procedure applies equally well to $\mathcal{E}$ - band (mixture) model spectral uncertainty classes as well, since they are a special case of the more general spectral-band model.

3.4. Direct Solution of the Least-Squares Non-Causal Problem

The preceeding section established a method of finding the state-by-state and least-squares robust solutions for non-causal, spectral-band model cases. An alternative means of determining least-squares non-causal solutions may be applied to all types of spectral classes, and leads to a slightly modified design procedure.

The least-squares maximin problem is stated

$$\max_{(S_\xi, S_\theta) \in \mathcal{X} \times \mathcal{N}} \left\{ \min_{H(j\omega) \in \mathcal{H}^n} E\,[(x-\hat{x})^T (x-\hat{x})] \right\}. \tag{3.4.1}$$

Equation (2.1.43) of Chapter 2 gives the error expression for the bracketed minimization,

$$\mathcal{E}_{opt}(S_\xi, S_\theta) = \frac{1}{2\pi} \operatorname{tr} \Big[\int_{-\infty}^{\infty} [I - H_{opt} G_c]\, S_x [I - H_{opt} G_c]^H + H_{opt} S_e H^H_{opt} \, d\omega \Big]. \tag{3.4.2}$$

Recalling that $G_x(\omega)$ is a vector (nx1), H_{opt} may be found.

$$S_{xy}(\omega) = G_x\, S_\xi\, G_y^H, \tag{3.4.3}$$

$$S_y(\omega) = |\, G_y\, |^2\, S_\xi + S_\theta \tag{3.4.4}$$

and therefore

$$H_{opt}(j\omega) = \frac{S_{xy}(\omega)}{S_y(\omega)} = G_x\, S_\xi\, G_y^H\, S_y^{-1}. \tag{3.4.5}$$

Also note that

$$S_x(\omega) = |\, G_x\, |^2\, S_\xi\ . \tag{3.4.6}$$

The error expression may then be expanded to

$$\mathcal{E}_{opt}(S_\xi, S_\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[tr(S_x) - tr(H_{opt}G_cS_x) - tr(S_xG_c^HH_{opt}^H) \\ + tr(H_{opt}G_cS_xG_c^HH_{opt}^H) + tr(H_{opt}S_\theta H_{opt}^H)]\, d\omega. \tag{3.4.7}$$

Computing each of the terms, applying the trace identity

$$tr\{AB\} = tr\{BA\}, \tag{3.4.8}$$

and noting that $tr\{D\} = D$ if D is a scalar,

i) $tr\{S_x\} = tr\{G_xS_\xi G_x^H\} = tr\{G_x^HG_xS_\xi\} = |G_x|^2\, S_\xi,$ (3.4.9)

ii) $tr\{H_{opt}G_cS_x\} = tr\{G_xS_\xi G_y^HS_y^{-1}G_cG_xS_\xi G_x^H\} = tr\{G_x^HG_xS_\xi G_y^HS_y^{-1}G_yS_\xi\}$

$$= |\, G_x|^2S_\xi[|G_y|^2S_\xi S_y^{-1}], \tag{3.4.10}$$

$$\text{iii)}\quad \mathrm{tr}\{S_x(H_{opt}G_c)^H\} = \mathrm{tr}\{G_x S_\xi G_x^H G_c^H S_y^{-H} G_y S_\xi^H G_x^H\} = |G_x|^2 S_\xi [G_y^H S_y^{-H} G_y S_\xi^H]$$

$$= |G_x|^2 S_\xi [|G_y|^2 S_\xi^H S_y^{-H}], \qquad (3.4.11)$$

$$\text{iv)}\quad \mathrm{tr}\{(H_{opt}G_c)S_x(H_{opt}G_c)^H\} = \mathrm{tr}\{G_x S_\xi G_y^H S_y^{-1} G_c G_x S_\xi G_x^H G_c^H S_y^{-H} G_y S_\xi G_x^H\}$$

$$= |G_x|^2 S_\xi \left[\frac{(|G_y|^2)^2 |S_\xi|^2}{|S_y|^2}\right], \qquad (3.4.12)$$

and

$$\text{v)}\quad \mathrm{tr}\{H_{opt} S_\vartheta H_{opt}^H\} = \mathrm{tr}\{G_x S_\xi G_y^H S_y^{-1} S_\vartheta S_y^{-H} G_y S_\xi^H G_x^H\}$$

$$= |G_x|^2 \frac{|S_\xi|^2 |G_y|^2}{|S_y|^2} S_\vartheta. \qquad (3.4.13)$$

The error may then be written

$$\mathcal{E}_{opt}(S_\xi, S_\vartheta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |1 - |G_y|^2 S_\xi S_y^{-1}|^2 \; |G_x|^2 S_\xi$$

$$+ \; |\; |G_y|^2 S_\xi S_y^{-1} \;|^2 \; \frac{|G_x|^2}{|G_y|^2} S_\vartheta \; d\omega. \qquad (3.4.14)$$

This is exactly the form of equation (3.3.1.2) when the tilde quantities are identified

$$\tilde{H}_{opt}(j\omega) = |G_y|^2 S_\xi S_y^{-1} \qquad (3.4.15)$$

$$\tilde{H}_{opt}(j\omega) = \frac{G_y G_x^H}{|G_x|^2} G_x G_y^H S_\xi S_y^{-1} \tag{3.4.16}$$

$$= \left[\frac{G_y G_x^H}{|G_x|^2}\right] H_{opt}(j\omega) \quad , \tag{3.4.17}$$

$$\tilde{S}_x(\omega) = \left[|G_x|^2\right] S_\xi \ , \tag{3.4.18}$$

$$\tilde{S}_\theta(\omega) = \left[\frac{|G_x|^2}{|G_y|^2}\right] S_\theta \ , \tag{3.4.19}$$

$$\tilde{\mathcal{X}} = \{ \tilde{S}_x : \tilde{S}_x = [\ |G_x|^2]\ S_\xi,\ \ S_\xi \in \mathcal{X} \} \ , \tag{3.4.20}$$

and

$$\tilde{\eta} = \{ \tilde{S}_\theta : \tilde{S}_\theta = \left[\frac{|G_x|^2}{|G_y|^2}\right] S_\theta,\ \ S_\theta \in \eta \}. \tag{3.4.21}$$

This observation leads to a direct method of solving least-squares robust filtering problems. Just as in Section 3.3.1, the maximin problem

$$\max_{(S_\xi, S_\theta) \in \mathcal{X} \times \eta} \left\{ \mathcal{E}_{opt}(S_\xi\ S_\theta) \right\} \tag{3.4.22}$$

is reformulated as

$$\max_{(\tilde{S}_x, \tilde{S}_\theta) \in \tilde{\mathcal{X}} \times \tilde{\eta}} \left\{ \tilde{\mathcal{E}}_{opt}(\tilde{S}_x, \tilde{S}_\theta) \right\}. \tag{3.4.23}$$

Since the two problems are equivalent, a saddlepoint solution to the tilde

problem (equation (3.4.23)) transforms to a saddlepoint solution to the original maximin problem (equation (3.4.22)) which is also a solution to the minimax problem (equation (3.2.5)) by Theorem 2.1. This simple transformation makes the solution of multiple-state least-squares problems possible whenever solutions have been found for the corresponding (tilde) one-dimensional problems.

Rapprochement of this method with the results of Subsection 3.3.2 concerning spectral-band models is straightforward. Equations (3.3.2.1) through (3.3.2.19) all hold if the vector G_x is substituted for the scalar G_{x_i}, so the transformations of equations (3.4.17)-(3.4.21) yield exactly the same results as the transformations of equations (3.3.1.6)-(3.3.1.10), as previously established in Theorem 3.1.

Section 3.3 began by considering the state-by-state formulation. In fact all of Subsection 3.3.1 is valid for any spectral classes for which the resulting tilde problem may be solved. However, the fact that the least favorable spectral pair is identical for each maximin problem of the state-by-state formulation has only been established for spectral-band problems (which include mixture model problems). Extension to other spectral uncertainty classes is not immedeate, and further research might establish that Theorem 3.1 also applies to other spectral classes (such as the total variation model) or perhaps to any classes of spectral density functions. It is conceivable then, that problems exist for which the least favorable spectral pair is not identical for each state of the state-by-state approach, and the state-by-state and least-squares formulations yield different filters.

Before considering causal problems, it might be helpful to summarize

the non-causal findings. For non-causal problems, the state-by-state robust solution may be found by reformulating each of the n maximin problems in equation (3.2.4) using the transformation of equations (3.3.1.6) - (3.3.1.10). The least-squares solution may be found by using the method of Section 3.4. For problems in which the spectral uncertainty classes are spectral-band models, the two formulations yield identical solutions and a computational procedure is proposed in which a single saddlepoint solution to one maximin problem of the state-by-state formulation yields the optimal filter and least favorable spectra.

3.5. Causal Problems

It would be highly desirable to establish robust state-estimation procedures for causal estimators. In this section, it is demonstrated that the methods used for non-causal design are unproductive in causal situations.

Recall the one-dimensional causal filter robust formulation

$$\max_{(\tilde{S}_x,\tilde{S}_\theta)\,\in\,\tilde{\mathcal{X}}\times\tilde{\mathcal{N}}} \left\{ \min_{\tilde{H}(j\omega)\,\in\,\mathcal{H}^\dagger} \frac{1}{2\pi}\int_{-\infty}^{\infty} |1 - \tilde{H}(j\omega)|^2 \tilde{S}_x + |\tilde{H}(j\omega)|^2 \tilde{S}_\theta \, d\omega \right\}. \tag{3.5.1}$$

Solution of the minimization yielded

$$\max_{(\tilde{S}_x,\tilde{S}_\theta)\,\in\,\tilde{\mathcal{X}}\times\tilde{\mathcal{N}}} \left\{ \frac{1}{2\pi}\int_{-\infty}^{\infty} |1 - \tilde{H}^\dagger(j\omega)|^2 \tilde{S}_x + |\tilde{H}^\dagger(j\omega)|^2 \tilde{S}_\theta d\omega \right\} \tag{3.5.2}$$

where

$$H^\dagger(j\omega) = \frac{1}{S_y^+(\omega)} \left[\frac{S_{xy}(\omega)}{S_y^-(\omega)} \right]_+ . \tag{3.5.3}$$

Following the derivation of Section 3.3, again each maximin problem of a state-by-state robust formulation may be written

$$\max_{(S_\xi, S_\theta) \in \mathcal{X} \times \mathcal{N}} \left\{ \min_{H_i(j\omega) \in \mathcal{H}^\dagger} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|1 - \frac{G_y}{G_{x_i}} H_i\right|^2 |G_{x_i}|^2 S_\xi + \left| \frac{G_y}{G_{x_i}} H_i \right|^2 \frac{|G_{x_i}|^2}{|G_y|^2} S_\theta \, d\omega \right\}, \tag{3.5.4}$$

and the transformations (3.3.6)-(3.3.10) made to yield a maximization identical in form to equation (3.5.1). The problem with the transformation arises due to the identity

$$\tilde{H}(j\omega) \triangleq \frac{G_y}{G_{x_i}} H_i(j\omega). \tag{3.5.5}$$

The causal Wiener filter result of equation (3.5.3) determines the causal $\tilde{H}(j\omega)$ which minimizes the cost function, whereas $H_i(j\omega)$ is the function which must be causal. Clearly, the transformation from equation (3.5.4) to (3.5.1) does not yield an equivalent problem and the solution to (3.5.1) is not generally a solution to (3.5.4).

Following this same reasoning, the direct least-squares approach of Section 3.4 is also invalid through the transformation and cannot be used.

Therefore, the design of causal multiple-state robust filters cannot be accomplished without significantly more elaborate techniques than presented here.

3.6. Error Bounds for Multiple-State Robust Estimators

Now that some useful procedures have been developed (at least for non-causal filters), the error performance may be analyzed for the established designs. The minimax formulation is intended to minimize over all possible filter structures the worst-case error. The best measure of system performance is naturally based on the worst-case estimation error.

Consider first the general non-causal, problem. Define the following rms errors

$$e_i = [\, E\, \{(x_i - \hat{x}_i)^2\}]^{1/2} \tag{3.6.1}$$

$$e = \begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ e_n \end{bmatrix} \tag{3.6.2}$$

e_i corresponds to the rms error in estimating the ith state. When the state-by-state formulation is used, an upper bound e_i^R can be found by solution of the ith maximin problem for each of the n states. The vector error e is then bounded in n-dimensional space by the hyperplanes $e_i = 0$ and $e_i = e_i^R$ for $i \in [1, \ldots, n]$. When the least-squares approach is used, the single maximin problem gives only a bound on the magnitude of the vector e, which will be denoted $|\, e\, |_{LS}^R$.

Whenever the least favorable spectral pair is identical for each

of the states using the state-by-state formulation as in spectral-band problems, the robust spectral pair yields the error vector

$$e^R = \begin{bmatrix} e_1^R \\ e_2^R \\ \cdot \\ \cdot \\ e_n^R \end{bmatrix} \tag{3.6.3}$$

In this case, the hyperrectangular region which forms the error region for the state-by-state filter is inscribed in the hyperspheric region of radius $|e|_{LS}^R$. This phenomenon is illustrated in two dimensions in Figure 3.1. Theorem 3.1 established clearly that for such a case the robust solutions are identical for state-by-state and least-squares methods. The obvious conclusion is that for problems which have the same least-favorable spectral solution for each maximin problem of the state-by-state approach, the state-by-state method produces a tighter error bound,, even though the filters are identical.

If the least-favorable spectra are not identical for each state's maximin problem, different filter structures will be generated for each of the two approaches and the only restrictions on the relationship between the two error regions are

$$\|e\|_{LS}^R \geq e_i^R \ , \ \forall i \ , \tag{3.6.4}$$

and by the least-squares optimality of $\|e\|_{LS}^2$

$$\|e\|_{LS}^R < \|e^R\| \ . \tag{3.6.5}$$

Such a case is illustrated in Figure 3.2.

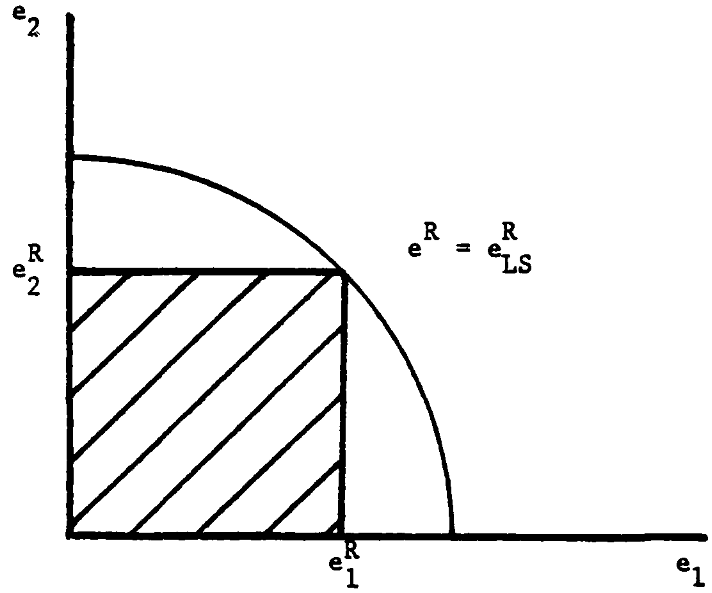

Figure 3.1 Spectral Band Error Bound

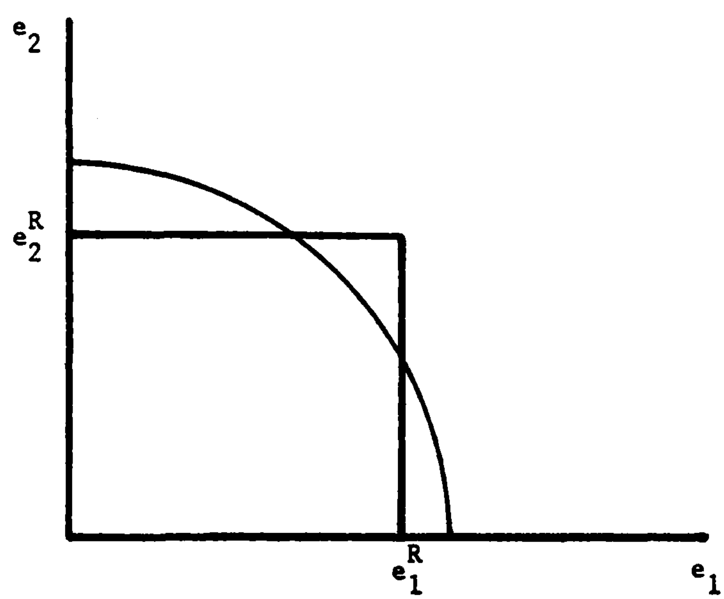

Figure 3.2 General Error Bound

4. AN ILLUSTRATIVE EXAMPLE - THE DOUBLE-INTEGRATOR PLANT WITH SPECTRAL-BAND UNCERTAINTY

The preceeding chapter developed techniques for designing robust multiple-state filters. The most productive problems for consideration using the concepts of Chapter 3 are non-causal and have noise process spectral uncertainty specified by spectral-band models. Accordingly, the following example is worked through in hopes of clarifying the application of the multiple-state methods.

<u>Example 4.1</u> Double-Integrator Plant, White Observation Noise of Spectral Height S_θ, Process Noise in Acceleration Only and Given by Spectral Band Model

The problem is stated in state-space form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xi \tag{4.0.1}$$

$$y = [\,1 \quad 0\,] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \theta \tag{4.0.2}$$

θ is white and of spectral height S_θ. ξ has spectrum $S_\xi(\omega)$ such that

$$S_\xi(\omega) \in \mathcal{X} \left\{ S_\xi(\omega) \; : \; S_L(\omega) \le S_\xi(\omega) \le S_U(\omega), \; \frac{1}{2\pi} \int_{-\infty}^{\infty} S_\xi(\omega) d\omega = 1 \right\} \tag{4.0.3}$$

where

$$S_L(\omega) = \frac{1}{\omega^2 + 1} \tag{4.0.4}$$

and

$$S_U(\omega) = \frac{3}{\omega^2 + 1} . \tag{4.0.5}$$

Identify the relevant transfer functions for solution:

$$G_{x_1} = \frac{-1}{\omega^2} , \tag{4.0.6}$$

$$G_{x_2} = \frac{1}{j\omega} , \tag{4.0.7}$$

$$G_x = \begin{bmatrix} -\frac{1}{\omega^2} \\ \\ \frac{1}{j\omega} \end{bmatrix} \tag{4.0.8}$$

and

$$G_y = \frac{-1}{\omega^2} . \tag{4.0.9}$$

To solve the least-squares problem directly, equations (3.4.17) through (3.4.19) give

$$\tilde{H}_{opt}(j\omega) = \left[\frac{G_y G_x^H}{|G_x|^2} \right] H_{opt} = [\ 1 - j\omega\ ]\ \frac{H_{opt}}{\omega^2 + 1} , \tag{4.0.10}$$

$$\tilde{S}_x(\omega) = |\ G_x\ |^2 S_\xi = \left[\frac{\omega^2 + 1}{\omega^4} \right] S_\xi \tag{4.0.11}$$

and

$$\tilde{S}_\theta(\omega) = \frac{|G_x|^2}{|G_y|^2}\ S_\theta = [\ \omega^2 + 1\ ] S_\theta . \tag{4.0.12}$$

By case a of Theorem A.1, with $N_L = N_U = S_\theta$,

$$S_\xi^R(\omega) = \begin{cases} S_L = \dfrac{1}{\omega^2+1}, & \dfrac{1}{\omega^2+1} > K_S\omega^4 S_\theta \\ K_S\omega^4 S_\theta, & \dfrac{1}{\omega^2+1} \le K_S\omega^4 S_\theta \le \dfrac{3}{\omega^2+1} \\ S_U = \dfrac{3}{\omega^2+1}, & \dfrac{3}{\omega^2+1} < K_S\omega^4 S_\theta \end{cases} \quad (4.0.13$$

as in equation (3.3.15). If the power constraint, P_ξ on the process noise is unity, K_S must be chosen to satisfy

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_\xi^R(\omega)d\omega = 1 \quad (4.0.14)$$

That is

$$\frac{1}{\pi}\int_0^{\omega_1}\frac{1}{\omega^2+1}d\omega + \frac{1}{\pi}\int_{\omega_1}^{\omega_2}K_S S_\theta \omega^4 d\omega + \frac{1}{\pi}\int_{\omega_2}^{\infty}\frac{3}{\omega^2+1}d\omega = 1 \quad (4.0.15)$$

where ω_1 is given by

$$\frac{1}{\omega^4(\omega^2+1)} = K_S S_\theta, \quad (4.0.16)$$

and ω_2 by

$$\frac{3}{\omega^4(\omega^2+1)} = K_S S_\theta. \quad (4.0.17)$$

Performing the integrations,

$$\frac{1}{\pi}[\arctan \omega_1] + \frac{K_S S_\theta}{5\pi}[\omega_2^5 - \omega_1^5] + \frac{3}{\pi}[\frac{\pi}{2} - \arctan \omega_2] = 1. \quad (4.0.18)$$

Solving (4.0.16)-(4.0.18) by iteration yields

$$K_S S_\theta = 1, \tag{4.0.19}$$

$$\omega_1 = .869, \tag{4.0.20}$$

$$\omega_2 = 1.084. \tag{4.0.21}$$

The final solution is then

$$S_\xi^R(\omega) = \begin{cases} \dfrac{1}{\omega^2+1}, & \omega < .869 \\[2ex] \omega^4, & .869 \le \omega \le 1.084 \\[2ex] \dfrac{3}{\omega^2+1}, & \omega > 1.084 \end{cases} \tag{4.0.22}$$

and by equation (3.3.17),

$$H_1^R(j\omega) = \begin{cases} \dfrac{1}{S_\theta\omega^6 + S_\theta\omega^4 + 1}, & \omega < .869 \\[2ex] \dfrac{K_S}{K_S + 1}, & .869 \le \omega \le 1.084 \\[2ex] \dfrac{3}{S_\theta\omega^6 + S_\theta\omega^4 + 1}, & \omega > 1.084 \end{cases} \tag{4.0.23}$$

and

$$H_2^R(j\omega) = \begin{cases} \dfrac{j\omega}{S_\theta\omega^6 + S_\theta\omega^4 + 1}, & \omega < .869 \\[2ex] \dfrac{j\omega K_S}{K_S + 1}, & .869 \leq \omega \leq 1.084 \\[2ex] \dfrac{3j\omega}{S_\theta\omega^6 + S_\theta\omega^4 + 1}, & \omega > 1.084 \end{cases} \qquad (4.0.24)$$

As discussed in Section 3.6, the tighter error bound is determined by using the robust solution in each state's maximin problem of the state-by-state formulation. The worst-case errors are most easily computed through the state-by-state tilde problems.

$$(e_i^R)^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |1 - \tilde{H}_i^R|^2\, \tilde{S}_{x_i} + |\tilde{H}_i^R|^2\, \tilde{S}_{\theta_i}\, d\omega \qquad (4.0.25)$$

$$(e_i^R)^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\tilde{S}_{x_i}^R\, \tilde{S}_{\theta_i}^R}{\tilde{S}_{x_i}^R + \tilde{S}_{\theta_i}^R}\, d\omega \qquad (4.0.26)$$

Using equations (3.3.7) and (3.3.8), first for $G_{x_1} = -\dfrac{1}{\omega^2}$

$$\tilde{S}_{x_1}^R = \begin{cases} \dfrac{1}{\omega^4(\omega^2 + 1)}, & \omega < .869 \\[2ex] 1, & .869 \leq \omega \leq 1.084 \\[2ex] \dfrac{3}{\omega^4(\omega^2 + 1)}, & \omega > 1.084 \end{cases} \qquad (4.0.27)$$

and

$$\tilde{S}^R_{\theta_1} = S_\theta = 1 , \tag{4.0.28}$$

and then for $G_{x_2} = \frac{1}{j\omega}$

$$\tilde{S}^R_{x_2} = \begin{cases} \dfrac{1}{\omega^2(\omega^2+1)}, & \omega < .869 \\ \omega^2, & .869 \le \omega \le 1.084 \\ \dfrac{3}{\omega^2(\omega^2+1)}, & \omega > 1.084 \end{cases} \tag{4.0.29}$$

and

$$\tilde{S}^R_{\theta_2} = \omega^2 S_\theta = \omega^2. \tag{4.0.30}$$

Putting these quantities into equation (4.0.26) shows that for this problem

$$(e^R_1)^2 = (e^R_2)^2 = \frac{1}{\pi}\int_0^{.869} \frac{1}{\omega^6+\omega^4+1}\, d\omega + \frac{1}{\pi}\int_{.869}^{1.084} \frac{1}{2}\, d\omega + \frac{1}{\pi}\int_{1.084}^{\infty} \frac{3}{\omega^6+\omega^4+3}\, d\omega. \tag{4.0.31}$$

The solution of this integral gives numerical values for e^R_1 and e^R_2. When specifying performance of the estimator the tightest bound which may be specified is

$$|e_1| \le e^R_1 \tag{4.0.32}$$

$$|e_2| \le e^R_2 , \tag{4.0.33}$$

rather than the less restrictive bound which results from the least squares method

$$e_1^2 + e_2^2 < (e_1^R)^2 + (e_2^R)^2. \tag{4.0.34}$$

5. CONCLUSION

5.1. Summary

The purpose of this paper was to investigate methods for applying results from current single-dimensional Wiener filtering theory to multivariable dynamical systems. To accomplish this end, consideration was restricted to systems with one observation process and multiple state processes, with the observation process corrupted by an additive scalar random process of uncertain spectral density, and the vector state process subject to a similar additive scalar noise.

Under this restriction to multiple-state systems several useful results were obtained. Two reasonable game-theoretic minimax formulations were suggested for the robust estimator. The state-by-state formulation minimizes each states worst case error over the allowable filter structures, while the least squares formulation minimizes the sum of the squared error of the states over the allowable (vector) filter structures. For non-causal problems, methods were derived for solving each of the formulations in terms of well-established one-dimensional results in Sections 3.3 and 3.4. Section 3.3.2 established that when each state problem of the state-by-state formulation has identical least-favorable spectral pair solutions, the state-by-state and least-squares robust solutions are identical. Problems of the spectral-band uncertainty type were shown to have this property. Additionally, in such cases, more restrictive bounds may be placed on the error using the state-by-state approach, even if the least-squares method seems more physically relevant to the design. All of these findings were based on simple transformations and known one-dimensional results.

Section 3.5 discussed briefly the failure of similar methods when a causal solution is desired.

Finally a design example was solved by the direct least-squares method of Section 3.4, and an error bound computed using the results of Section 3.3.2, to illustrate an application of the theoretical results of Chapter 3.

5.2. Topics for Further Research

Due to the rather restrictive assumptions made in the bulk of this investigation, several issues for additional study are apparent from the outset.

Section 3.1 mentioned briefly the desirability of techniques for handling truly multivariable systems with vector observation and noise processes. It is anticipated that much of the work presented here is closely related to methods which could be productive for such problems.

The difficulties encountered in Section 3.5 for causal robust problems are primarily a result of the fact that the set of causal filter transfer functions is not invariant under the necessary transformation. These problems might be approached as outlined if more general one-dimensional conventional Wiener filtering results are derived, but the author suspects that an entirely different approach is more feasible.

Perhaps the most blatant question which remains unresolved concerns the class of problems to which Theorem 3.1 may be applied. Specifically, the types of problems for which each states least favorable spectral pair is identical was shown to include spectral-band

and mixture models. The same may be true in general for all spectral uncertainty models, or at least for a large variety of models. Establishment of such a fact produces immediate solutions using the results of this paper for non-causal multiple-state problems, so studies of this type might prove very worthwhile.

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APPENDIX

The somewhat lengthy theorem due to Kassam and Lim is presented here very much as it appears in their article [4], except for revision of notation to agree with that of this paper.

First define the following sub-sets of the real line, which will appear in the statements and proofs of the theorem for the band-model

$$\alpha_1(k_s) \triangleq \{\omega | S_L(\omega) < k_s N_L(\omega) \le S_U(\omega)\}, \quad (A.1)$$

$$\alpha_2(k_s) \triangleq \{\omega | S_L(\omega) \le S_U(\omega) < k_s N_L(\omega)\}, \quad (A.2)$$

$$\beta_1(k_n) \triangleq \{\omega | k_n N_L(\omega) \le S_L(\omega) < k_n N_U(\omega)\}, \quad (A.3)$$

and

$$\beta_2(k_n) \triangleq \{\omega | k_n N_L(\omega) \le k_n N_U(\omega) \le S_L(\omega)\}, \quad (A.4)$$

these sets being dependent on the values of the finite, positive parameters k_s and k_n. Also define the sets $\alpha(k_s)$, $\beta(k_n)$ by

$$\alpha(k_s) \triangleq \alpha_1(k_s) U \alpha_2(k_s) = \{\omega | S_L(\omega) < k_s N_L(\omega)\} \quad (A.5)$$

and

$$\beta(k_n) \triangleq \beta_1(k_n) U \beta_2(k_n) = \{\omega | k_n N_L(\omega) \le S_L(\omega)\}. \quad (A.6)$$

Also define the following four sets:

$$b_1(\ell_s) \triangleq \{\omega | S_U(\omega) > \ell_s N_U(\omega) \ge S_L(\omega)\}, \quad (A.7)$$

$$b_2(\ell_s) \triangleq \{\omega | S_U(\omega) \ge S_L(\omega) > \ell_s N_U(\omega)\}, \quad (A.8)$$

$$a_1(\ell_n) \triangleq \{\omega | \ell_n N_U(\omega) \ge S_U(\omega) > \ell_n N_L(\omega)\}, \quad (A.9)$$

$$a_2(\ell_n) \triangleq \{\omega | \ell_n N_U(\omega) \ge \ell_n N_L(\omega) \ge S_U(\omega)\}, \quad (A.10)$$

where the parameters ℓ_s, ℓ_n are finite and positive. Again define the sets $a(\ell_s)$ and $b(\ell_n)$ by $a(\ell_s) \triangleq a_1(\ell_s) \cup a_2(\ell_s)$ and $b(\ell_n) \triangleq b_1(\ell_n) \cup b_2(\ell_n)$, so that

$$b(\ell_s) \triangleq \{\omega | S_U(\omega) > \ell_s N_U(\omega)\} \tag{A.11}$$

and

$$a(\ell_n) \triangleq \{\omega | \ell_n N_U(\omega) \geq S_U(\omega)\}. \tag{A.12}$$

The structure of the robust solution to the filtering problem depends on the nature of, and existence of, the solutions of the following equations

$$P_s(x)|_{k_s} = 2\pi P_\xi \tag{A.13}$$

$$P_n(x)|_{k_n} = 2\pi P_\theta \tag{A.14}$$

where

$$P_s(x) = x \int_{\alpha_1(x)} N_L(\omega)d\omega + \int_{\alpha_2(x)} S_U(\omega)d\omega + \int_{\bar{\alpha}(x)} S_L(\omega)d\omega \tag{A.15}$$

and

$$P_n(x) = \frac{1}{x} \int_{\beta_1(x)} S_L(\omega)d\omega + \int_{\beta_2(x)} N_u(\omega)d\omega + \int_{\bar{\beta}(x)} N_L(\omega)d\omega. \tag{A.16}$$

In addition, for the band-model we may have to consider solutions for

$$Q_s(x)|_{\ell_s} = 2\pi P_\xi \tag{A.17}$$

$$Q_n(x)|_{\ell_n} = 2\pi P_\theta \tag{A.18}$$

where

$$Q_s(x) = x \int_{b_1(x)} N_U(\omega)d\omega + \int_{b_2(x)} S_L(\omega)d\omega + \int_{\bar{b}(x)} S_U(\omega)d\omega \tag{A.19}$$

and

$$Q_n(x) = \frac{1}{x} \int_{a_1(x)} S_U(\omega)d\omega + \int_{a_2(x)} N_L(\omega)d\omega + \int_{\bar{a}(x)} N_U(\omega)d\omega. \tag{A.20}$$

The robust solution for the band-model filtering problem can now be stated explicitly:

<u>Theorem 1</u> (Band-Model): $H^R(\omega)$ (satisfying equation (2.2.14))

The most robust filter, $H^R(\ )$ (satisfying equation (2.2.14)) for the pair of PSD's $S_\xi^R(\omega) \in \mathcal{X}$ and $S_\theta^R(\omega) \in \mathcal{N}$ defined according to one of the following:

(a) If $k_s - k_n$ exist satisfying (A.13) and (A.14) then

$$S_\xi^R(\omega) = \begin{cases} k_s N_L(\omega), & \omega \in \alpha_1(k_s) \\ S_U(\omega), & \omega \in \alpha_2(k_s), \\ S_L(\omega), & \omega \in \bar{\alpha}(k_s) \end{cases} \tag{A.21}$$

$$S_\theta^R(\omega) = \begin{cases} \frac{1}{k_n} S_L(\omega), & \omega \in \beta_1(k_n) \\ N_U(\omega), & \omega \in \beta_2(k_n). \\ N_L(\omega), & \omega \in \bar{\beta}(k_n) \end{cases} \tag{A.22}$$

(b) If (A.13) and (A.14) do not have solutions k_s, k_n satisfying $k_s \le k_n$, and if the following are valid PSD's for the band-model, then

$$S_\xi^R(\omega) = \begin{cases} kN_L(\omega) + S_e(\omega), & \omega \in \alpha_1(k) \\ S_U(\omega) & \omega \in \alpha_2(k) \\ S_L(\omega) + S_e(\omega), & \omega \in \beta_1(k) \\ S_L(\omega), & \omega \in \beta_2(k) \end{cases}, \tag{A.23}$$

$$S_\theta^R(\omega) = \begin{cases} \frac{1}{k} S_L(\omega) + N_e(\omega), & \omega \in \beta_1(k) \\ N_U(\omega), & \omega \in \beta_2(k) \\ N_L(\omega) + N_e(\omega), & \omega \in \alpha_1(k) \\ N_L(\omega) & \omega \in \alpha_2(k) \end{cases}, \tag{A.24}$$

where k satisfies

$$k = \frac{2\pi P_\xi - \int_{\beta_2(k)} S_L(\omega)d\omega - \int_{\alpha_2(k)} S_U(\omega)d\omega}{2\pi P_\theta - \int_{\beta_2(k)} N_U(\omega)d\omega - \int_{\alpha_2(k)} N_L(\omega)d\omega} \tag{A.25}$$

and $S_e(\omega)$, $N_e(\omega)$ can be any non-negative functions satisfying

$$S_e(\omega) = kN_e(\omega). \tag{A.26}$$

(c) If neither (a) nor (b) above yield the robust solution, then if ℓ_s, ℓ_n are solutions of (A.17) and (A.18) (with $\ell_s > \ell_n$ necessarily), we have

$$S_\xi^R(\omega) = \begin{cases} \ell_s N_U(\omega), & \omega \in b_1(\ell_s) \\ S_L(\omega), & \omega \in b_2(\ell_s), \\ S_U(\omega), & \omega \in \bar{b}(\ell_s) \end{cases} \tag{A.27}$$

$$S_\theta^R(\omega) = \begin{cases} \frac{1}{\ell_n} S_U(\omega), & \omega \in a_1(\ell_n) \\ N_L(\omega), & \omega \in a_2(\ell_n). \\ N_U(\omega), & \omega \in \bar{a}(\ell_n) \end{cases} \tag{A.28}$$

Finally, if (A.17) for ℓ_s has no solution, $S_\xi^R(\omega)$ can be picked to be $S_U(\omega)$ when $N_U(\omega) > 0$ and arbitrary otherwise, and if (A.18) for ℓ_n has no solution, $S_\theta^R(\omega)$ can be picked to be $N_U(\omega)$ when $S_U(\omega) > 0$ and arbitrary otherwise (satisfying the power constraints).

<u>Proof</u>: See [4].

END